Solving Constraint Satisfaction Puzzles with Constraint Programming

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Abstract

Constraint Programming (CP) is a powerful paradigm for solving Combinatorial Problems (generally issued from Decision Making). In CP, Enumeration Strategies are crucial for resolution performances. In this work, we model the known benchmark problems Latin Square, Magic Square and Sudoku as a Constraint Satisfaction Problems. We solve them with Constraint Programming comparing the performance of different Variable and Value Selection Heuristics in its Enumeration phase. The platform used was Mozart[1].

Keywords: Constraint Programming, Enumeration Strategies, Variable Selection Heuristics, Value Selection Heuristics, Combinatorial Problems.

Resumen

La Programación con Restricciones es un poderoso paradigma para resolver Problemas Combinatoriales (generalmente utilizados en Toma de Decisiones). Resolviendo este tipo de problemas, el rendimiento de la Programación con Restricciones depende crucialmente de la Estrategia de Enumeración. En este trabajo se modelan como Problemas de Satisfacción de Restricciones los conocidos problemas benchmark Cuadrados Latinos, Cuadrados Mágicos y Sudoku. Los problemas se resuelven con Programación con Restricciones comparando en su Fase de Enumeracón el rendimiento de diferentes Heurísticas de Seleccion de Variable y Valor. La implementación se hizo en Mozart.

Palabras Clave: Programación con Restricciones, Estrategias de Enumeración, Heurísticas de Seleccion de Variable, Heurísticas de Seleccion de Valor, Problemas Combinatoriales.

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1 Introduction

The Constraint Programming (CP) has been defined as a technology of Software used to describe and solve combinatorial problems [1]. The main idea of this paradigm is to model a problem by mean of a declaration of variables and constraints and to find solutions that satisfy all the constraints. Many of the combinatorial problems focused by CP can be modeled like a Constraint Satisfaction Problem (CSP), which consist of a sequence of variables $X = x_1, x_2, ..., x_n$ with its respective domains $D = D_{x_1}, D_{x_2}, ..., D_{x_n}$, and a finite set C of constraints restricting the values that the variables can take simultaneously. The goal is to assign a value to each variable satisfying all the constraints. The basic mechanism underlying CP to solve a CSP interleaves Constraint Propagation (network consistency) and Enumeration (distribution or labeling). In essence, the algorithm increases the efficiency of the search by looking ahead actively using the constraints to prune the search space. The underlying structure to the described the CP paradigm is shown in Figure 1.

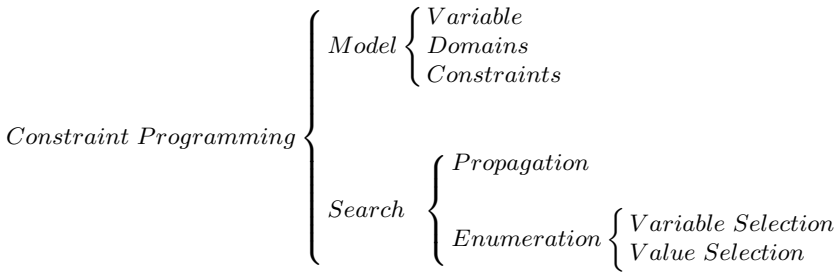

Fig. 1. Constraint Programming Approach

This work is focused on the Enumeration phase of CP, where the use of variable and value selection heuristics is critical. A suitable definition and use of an enumeration strategy can improve the resolution process strongly. We apply to resolution of puzzles (Magic Square, Latin Square and Sudoku) different variables and values selection heuristics presented in the literature [2].

2 Resolution Technique

In the resolution of Constraint Satisfaction Problems diverse techniques can be used, currently they are solved using complete techniques (global optimization), incomplete techniques (local optimization), and hibridizations of both techniques. Specifically, the Constraint Programming community uses a complete approach alternating phases of constraint propagation and enumeration, where the propagation prunes the search tree by eliminating values that can not participate in a solution. Enumeration [1] consists of dividing the

original CSP in two smaller CSPs, creating one branch by instantiating a varible ($x = v$) and another branch ($x \neq v$) for backtracking when the first branch does not contain any solution. When enumerating two decisions have to be made: What variable is selected to be instantiated? and What value is assigned to the selected variable?. In order to support these decisions we use enumeration strategies.

3 Enumeration Strategy

The enumeration strategies are constituted by variable and value selection heuristics [2].

3.1 Variable Selection Heuristics

The main idea that exists within the choice of the next variable, is to minimize the size of the search tree and to ensure that any branch that does not lead to a solution is pruned as early as possible, this was termed as the "fail-first" principle by Haralick and Elliot [3], described as *"To succeed, try first where you are most likely to fail"* [4, 1]. For variable selection, it is possible to find a classification according to the moment when the selection order is done.
Static Selection Heuristic: it generates a fixed order of the variables before initiating the search. Here the variables are always selected in the order predefined for instantiation.
Dynamic Selection Heuristic: it can change the instantiation order of the variables dynamically as one advances in the tree search. It is based on information generated during the search. Here the dynamic and static terms are used according to the definition given in [4].

In this work we used the following variable selection heuristics:
Minimum Domain Size (*MiD*): at each enumeration step the domain of each one of the variables not yet instantiated is analyzed, then the variable with smaller domain size is selected.
Maximum Domain Size (*MaD*): the idea of this heuristic is similar to the previous one, nevertheless in this case it selects the variable with the greater domain size.
Order Previously Established (*Opre*): in this static heuristic for magic squares, selecting first the elements of the main diagonal, after the elements in the upper triangle of the main diagonal and finally the elements in the lower triangle of the main diagonal.

3.2 Value Selection Heuristics

In choosing the value, we can try, if it is possible, a value which is likely to lead to a solution, and so reduce the risk of having to backtrack and try an alternative value ("succeed-first" principle[4]). In this work we used the following value selection heuristics:
Smaller Value of the Domain (*SVal*): this heuristic establishes that the smallest value of the domain is always chosen.

Greater Value of the Domain (*GVal*): it is similar to the previous one, but instead of choosing the smallest element of the domain, the greater element is selected.
Average Value of the Domain (*AVal*): this heuristic selects the value of the domain that is more near to the half of the domain, it calculates the arithmetic average between the limits (superior and inferior) of the domain of the selected variable and in case of having a tie the smallest value is selected.
Immediately Greater Value to the Average Value of the Domain (*GAV*): this heuristics selects the smaller value of the domain that it is greater as well to the average value of the domain. Finally, established the heuristic to use, the enumeration strategies are compound according to Table 1.

S_1 = MiD+SVal	S_5 = MaD+SVal	S_9 = Opre+SVal
S_2 = MiD+GVal	S_6 = MaD+GVal	S_{10} = Opre+GVal
S_3 = MiD+AVal	S_7 = MaD+AVal	S_{11} = Opre+AVal
S_4 = MiD+GAV	S_8 = MaD+GAV	S_{12} = Opre+GAV

Table 1. Enumeration Strategies

4 Description of Problems

4.1 Magic Square

This puzzle consists in finding for a given N an NxN matrix such that every cell of the matrix is a number between 1 and N^2, all the cells of the matrix must to be different, and the sum of the rows, columns, and the two diagonals are all equal. The mathematical representation used to model the problems has a variable x_{ij} that represents the value that each cell (i, j) of the matrix can take, and a variable S for the sum of each row, column and diagonal. Then the CP model establishes the following constraint:

$$\forall i, j \in \{1, ..., N\} \;\; Alldifferent\{x_{ij}\} \tag{1}$$

$$\sum_{j=1}^{N} x_{ij} = S \;\; \forall i \in \{1, ..., N\} \tag{2}$$

$$\sum_{i=1}^{N} x_{ij} = S \;\; \forall j \in \{1, ..., N\} \tag{3}$$

$$\sum_{i=1}^{N} x_{ii} = S \tag{4}$$

$$\sum_{i=1}^{N} x_{i(N-i+1)} = S \tag{5}$$

The constraints (2) and (3) ensure that the sum of each row and each column will be equal to S, and the constraints (4) and (5) assure that the sum of each diagonal will be equal to S.

4.2 Latin Square

A Latin Square puzzle of order N is defined as an NxN matrix where all its elements are numbers between 1 and N with the property that each one of the N numbers appear exactly once in each row and exactly once in each column of the matrix. The CP model consists of the following constraints:

$$\forall i \in \{1, ..., N\} \quad Alldifferent\{x_{i1}, x_{i2}, ..., x_{iN}\} \tag{6}$$

$$\forall j \in \{1, ..., N\} \quad Alldifferent\{x_{1j}, x_{2j}, ..., x_{Nj}\} \tag{7}$$

4.3 Sudoku

Sudoku is a puzzle played in a 9x9 matrix (standard sudoku) which, at the beginning, is partially full. This matrix is composed of 3x3 submatrices denominated "regions". The task is to complete the empty cells so that each column, row and region contain numbers from 1 to 9 exactly once [5]. The model used for the representation can be seen like a composition of the models used in the above puzzles:

$$\forall i \in \{1, ..., 9\} \quad Alldifferent\{x_{i1}, x_{i2}, ..., x_{i9}\} \tag{8}$$

$$\forall j \in \{1, ..., 9\} \quad Alldifferent\{x_{1j}, x_{2j}, ..., x_{9j}\} \tag{9}$$

On the other hand, each cell in regions S_{kl} with $0 \leq k, l \leq 2$ must be different, which forces to include in the model the following constraint:

$$\forall i, j \quad Alldifferent\{x_{ij}, x_{i(j+1)}, x_{i(j+2)}, x_{(i+1)j}, \tag{10}$$

$x_{(i+1)(j+1)}, x_{(i+1)(j+2)}, x_{(i+2)j}, x_{(i+2)(j+1)}, x_{(i+2)(j+2)}\}$

con $i = k * 3 + 1$ y $j = l * 3 + 1$.

5 Analysis of Results

Each one of the exposed problems were implemented and solved in the platform Mozart with the eight first strategies listed in Table 1, and additionally the strategies $S_9, ..., S_{12}$ were used in the resolution of magic squares puzzles, because these strategies are constituted by a variable selection heuristic designed specifically for such problem. Each execution had a time limited to 10 minutes, not finding results are indicated with the symbol "-". The tests conducted allow to evaluate the performance of the enumeration strategies based on the following indicators of performance:

Number of Backtracking (B): it shows the amount of bad decisions made during the search of the solution, that is calculations or decisions executed without leading to a solution.

Number of Enumerations (E): this metric tells the amount of nodes or spaces generated to find the solution of the problem, including the good enumerations that lead to a solution and the bad enumerations that force to backtrack.

Time (t): it measures the required time to solve the problem.

5.1 Searching the First Solution

In general it is possible to appreciate that for small instances the strategies do not reflect significant differences. Nevertheless when observing the results obtained for each one of the problems it is possible to see that the performance of the strategies varies as N increases, this because with the increase of N the size of the search space grows drastically. On the other hand, when observing the results obtained it is perceived that the strategies constituted by the heuristic *MiD* $(S_1, ..., S_4)$ have better behavior in those instances in which the search space grows, this in comparison with strategies that are guided by the heuristic *MaD* $(S_5, ..., S_8)$. Such differences happen mainly because the heuristic *MiD* leads as rapidly as possible to an insolvent space, allowing to prune the tree search. Leading to an insolvent space quickly consists of choosing variables with few elements in its domain, increasing the probability of failing before generating a big search tree.

N	3			4			5			10			15		
	(E)	(B)	(t)	(E)	(B)	(t)	(E)	(B)	(t)	(E)	(B)	(t)	(E)	(B)	(t)
S_1	3	0	9	6	0	10	10	0	10	67	1	18	165	7	76
S_2	3	0	10	6	0	10	10	0	10	67	1	18	165	7	78
S_3	3	0	10	6	0	10	12	0	12	70	2	18	163	5	67
S_4	3	0	10	6	0	10	10	0	11	70	4	31	309	138	229
S_5	3	0	12	8	0	10	94	77	16	-	-	-	-	-	-
S_6	3	0	9	8	0	10	94	77	15	-	-	-	-	-	-
S_7	3	0	10	8	0	10	1644	1625	106	-	-	-	-	-	-
S_8	3	0	10	8	0	10	36	21	12	-	-	-	-	-	-

Table 2. Latin Squares: Enumerations (E), Backtracking (B), CPU time (t) in ms.

Observing the results obtained for the magic square problem, one of the aspects to emphasize is the good performance obtained by the strategies constituted by the heuristic *Opre*, particularly using it in combination with value selection heuristic *AVal* and *GAV*, this good behavior reflected in the Table 3, it is due mainly because the heuristic *Opre* generates a more effective constraint propagation, reducing the size of the search space considerably. The reduction of the search space takes place because at the beginning of the process the variables chosen are related to a greater number of other variables (elements of the main diagonal), which produces that when evaluating the levels of consistency between the variables is eliminated a greater amount of inconsistency, reducing the space to explore. In order to conclude, it is possible to mention that the size of the search space has a great incidence in the resolution process, where his exponential growth makes the process considerably more expensive, this is possible to appreciate when solving different instances of the same problem (of course, hypothesis demonstrated in a lot of previous work) With regard to the

N^2	9			16			25			49		
	(E)	(B)	(t)	(E)	(B)	(t)	(E)	(B)	(t)	(E)	(B)	(t)
S_1	4	1	1	741	717	30	208861	208827	12669	-	-	-
S_2	13	7	1	1681	1655	63	9113251	9113227	505231	-	-	-
S_3	8	5	1	2465	2449	81	2187	2158	76	101510	101424	4415
S_4	13	7	1	406	384	27	7061	7037	452	7139	7081	487
S_5	16	7	1	45097	45043	3680	-	-	-	-	-	-
S_6	14	8	1	681	646	46	-	-	-	-	-	-
S_7	22	10	1	847094	847055	48519	-	-	-	-	-	-
S_8	23	19	1	358775	358772	27115	557090	557608	44189	-	-	-
S_9	5	2	0	49	34	3	26475	26428	1396	-	-	-
S_{10}	13	7	1	508	491	24	80761	80728	5078	-	-	-
S_{11}	8	5	1	1323	1296	56	393	375	21	-	-	-
S_{12}	13	7	1	791	763	55	12	1	2	82107	82047	6649

Table 3. Magic Squares: Enumerations (E), Backtracking (B), CPU time (t) in ms

		S_1			S_2			S_3			S_4		
Source	Degree	(E)	(B)	(t)	(E)	(B)	(t)	(E)	(B)	(t)	(E)	(B)	(t)
SudokuMin	None-1	84	52	14	220	195	21	1308	1283	88	183	159	26
SudokuMin	None-2	2836	2815	153	271	249	23	11074	11048	603	124	102	22
The Times	Easy	7	3	11	17	13	11	7	3	10	17	13	12
The Times	Medium	16	6	11	174	164	19	16	6	11	174	164	26
The Times	Hard	27	16	11	24	18	11	27	16	11	24	18	12

Table 4. Sudoku solved with heuristic *MiD*

		S_5			S_6		
Source	Degree	(E)	(B)	(t)	(E)	(B)	(t)
SudokuMin	None-1	-	-	-	-	-	-
SudokuMin	None-2	-	-	-	-	-	-
The Times	Easy	18554	18537	1799	274476	274472	28149
The Times	Medium	-	-	-	121135	121113	12868
The Times	Hard	-	-	-	-	-	-

Table 5. Sudoku solved with S_5 and S_6 strategy

		S_7			S_8		
Source	Degree	(E)	(B)	(t)	(E)	(B)	(t)
SudokuMin	None-1	-	-	-	-	-	-
SudokuMin	None-2	-	-	-	-	-	-
The Times	Easy	24195	24169	2582	721773	72155	7484
The Times	Medium	-	-	-	88720	88706	9763
The Times	Hard	93138	93105	9158	-	-	-

Table 6. Sudoku solved with S_7 and S_8 strategy

resolution of Sudoku [5], different published instances have been used from The Times[2] and Minimum Sudoku page[3].

In Table 4, Table 5 and Table 6 we show the source from where puzzles were obtained, the degree (difficulty level), the heuristic used to solve each instance and the different measured indicators of performance during the execution of the tests. Although some authors say that the amount of numbers given initially does not have incidence in the degree of difficulty of Sudoku, the results obtained here show bad results with 17 numbers in comparison with the other cases (easy, medium and hard), where the amount of numbers given initially is greater 25.

6 Conclusions

In this work we showed that variable and value selection heuristics influence the efficiency in the resolution of combinatorial problems. The efficiency of resolution was measured on the basis of performance indicators. The work included the modeling and resolution of classic puzzles (Magic Square, Latin Square and Sudoku) in Mozart. The possibility to obtain better results in the search process was showed using suitable criteria of selection of variables and values. In fact, to select a variable in a search process implies to determine the descending nodes of the present space that have a solution. It is very important to detect when the descending nodes are not in a solution, because in this way we avoided to do unnecessary calculations that force to backtracking. Due to the above reasoning the heuristic selecting the variable with minimum domain size (MiD) presents a better behavior in comparison with the other strategies, because MiD bets by the variable going quickly towards an insolvent space avoiding a priori unnecessary calculations. We showed that the resolution possibilities of a certain problem depends on the search space size, it is possible to be appreciated when observing the differences generated in the results obtained in the resolution of different instances of the same problem.

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